Exercise Answer the following questions.

a. Suppose a function f is differentiable and its derivative f' is strictly decreasing.
Show that the following inequalities hold:

$$f'(x+1) < f(x+1) - f(x) < f'(x).$$

b. Consider two sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ defined by

$$x_n = \left(1+\frac{1}{n}\right)^n \quad \text{and} \quad y_n = \left(1+\frac{1}{n}\right)^{n+1}$$

Show that ① $x_n < x_{n+1} < e < y_{n+1} < y_n$ for all $n \in \mathbb{N}$

and ② $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = e$

c. By using the results of b, show that

$$\left(\frac{n}{e}\right)^n < n! < en\left(\frac{n}{e}\right)^n \quad \text{for all } n\in\mathbb{N}.$$

d. Show that $\lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n} = \frac{1}{e}$

Solution.

a. By mean value theorem, for any x, there exists c between x and $x+1$ such that

$$f'(c) = \frac{f(x+1)-f(x)}{(x+1)-x} = f(x+1) - f(x).$$

But, since the derivative function f' is strictly decreasing,
$f'(x+1) < f'(c) < f'(x)$.
Hence, $f'(x+1) < f(x+1) - f(x) < f'(x)$. ▨

b. ① Let $\begin{cases} g(x) = x\ln\left(\frac{x+1}{x}\right) = x(\ln(x+1) - \ln x), \\ h(x) = (x+1)\ln\left(\frac{x+1}{x}\right) = (x+1)(\ln(x+1) - \ln x). \end{cases}$

Then, $\begin{cases} x_n = e^{g(n)} \\ y_n = e^{h(n)} \end{cases}$ for all $n\in\mathbb{N}$.

Now, let us consider a function $f(x) = \ln x$.
Since its derivative $f'(x) = \frac{1}{x}$ is strictly decreasing, one can apply a. to f. Hence we have

$$\frac{1}{x+1} < \ln(x+1) - \ln x < \frac{1}{x}.$$

(b. continued)

This shows that, for $x > 0$,

$g(x) = x(\ln(x+1) - \ln x) < 1$ and $h(x) = (x+1)(\ln(x+1) - \ln x) > 1$.

Hence, for any $n \in \mathbb{N}$, we have

$x_n = e^{g(n)} < e$ and $y_n = e^{h(n)} > 1$.

Furthermore, for $x > 0$, we have

$g'(x) = \ln(x+1) - \ln x - \frac{1}{x+1} > 0$

and $h'(x) = \ln(x+1) - \ln x - \frac{1}{x} < 0$.

Hence, the function $\begin{cases} g \text{ is increasing,} \\ h \text{ is decreasing.} \end{cases}$

It follows that, for any $n \in \mathbb{N}$,

$x_n = e^{g(n)} < e^{g(n+1)} = x_{n+1}$ and

$y_n = e^{h(n)} > e^{h(n+1)} = y_{n+1}$.

Finally, we have $x_n < x_{n+1} < e < y_{n+1} < y_n$ for all $n \in \mathbb{N}$.

② Because the sequence $\{x_n\}$ is increasing and bounded above, it converges to some number by monotone convergence theorem.

Similarly, because the sequence $\{y_n\}$ is decreasing and bounded below, it converges.

But, since $\lim_{n\to\infty} \frac{y_n}{x_n} = \lim_{n\to\infty} \frac{(1+\frac{1}{n})^{n+1}}{(1+\frac{1}{n})^n} = \lim_{n\to\infty}(1+\frac{1}{n}) = 1$, we have

$\lim_{n\to\infty} y_n = \lim_{n\to\infty}\left(\frac{y_n}{x_n} \cdot x_n\right) = \lim_{n\to\infty}\frac{y_n}{x_n} \lim_{n\to\infty} x_n = 1 \cdot \lim_{n\to\infty} x_n = \lim_{n\to\infty} x_n$.

But, from ①, we have observed that

$\lim_{n\to\infty} x_n \le e$ and $\lim_{n\to\infty} y_n \ge e$.

Hence, $\lim_{n\to\infty} x_n = e = \lim_{n\to\infty} y_n$. $\boxtimes$

C. First we show that $\left(\frac{n}{e}\right)^n < n!$.

Observe $\left(\frac{n}{e}\right)^n / \left(\frac{n-1}{e}\right)^{n-1} = \frac{n}{e}\left(\frac{n}{n-1}\right)^{n-1} = \frac{n}{e} \cdot x_{n-1} < \frac{n}{e} \cdot e = n$ for all $n \in \mathbb{N}$

by b.

Hence, we have

$\left(\frac{n}{e}\right)^n = \left(\frac{n}{e}\right)^n / \left(\frac{n-1}{e}\right)^{n-1} \cdot \left(\frac{n-1}{e}\right)^{n-1} / \left(\frac{n-2}{e}\right)^{n-2} \cdot \;\cdots\; \cdot \left(\frac{1}{e}\right)^1 / \left(\frac{0}{e}\right)^0$

$< n \cdot (n-1) \cdot \;\cdots\; 1 = n!$

(c. continued)

Now, we show that $n! < en\left(\frac{n}{e}\right)^n$.

$$\left(\frac{n}{e}\right)^{n+1} \Big/ \left(\frac{n-1}{e}\right)^n = \frac{n}{e}\left(\frac{n}{n-1}\right)^n = \frac{n}{e} y_{n-1} \underset{b.}{>} \frac{n}{e}\cdot e = n \quad \text{for all } n \in \mathbb{N}_{\geq 2}.$$

Hence we have

$$\left(\frac{n}{e}\right)^{n+1} \Big/ \left(\frac{1}{e}\right)^2 = \left(\left(\frac{n}{e}\right)^{n+1} \Big/ \left(\frac{n-1}{e}\right)^n\right)\cdot\left(\left(\frac{n-1}{e}\right)^n \Big/ \left(\frac{n-2}{e}\right)^{n-1}\right) \cdots \left(\left(\frac{2}{e}\right)^3 \Big/ \left(\frac{1}{e}\right)^2\right)$$

$$> n \cdot (n-1) \cdot \cdots \cdot 2 = n!$$

$$\Rightarrow \left(\frac{n}{e}\right)^{n+1}\cdot e^2 = en\left(\frac{n}{e}\right)^n > n! \quad ▨$$

d. By c, we have, for any $n \in \mathbb{N}$,

$$\left(\frac{n}{e}\right)^n < n! < ne\left(\frac{n}{e}\right)^n \quad \text{for all } n \in \mathbb{N}$$

$$\Rightarrow \left(\frac{1}{e}\right)^n < \frac{n!}{n^n} < ne\left(\frac{1}{e}\right)^n$$

$$\Rightarrow \frac{1}{e} < \sqrt[n]{\frac{n!}{n^n}} = \frac{\sqrt[n]{n!}}{n} < \frac{\sqrt[n]{ne}}{e}$$

Since $\lim_{n\to\infty} \frac{\sqrt[n]{ne}}{e} = \frac{1}{e}$, by sandwich theorem, we have $\lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n} = \frac{1}{e}$. ▨